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Existence of normal bimagic squares

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ABSTRACT

In this paper we provide a construction of normal bimagic squares by means of a magic pair of orthogonal general bimagic squares. It is shown that a normal bimagic square of order mn exists for all positive integers m, n such that $m, n \notin \{2, 3, 6\}$ and $m \equiv n \pmod 2$, and a normal bimagic square of order $4m$ exists if and only if $m \geq 2$.

1. Introduction

An $n \times n$ matrix A consisting of nonnegative integers is a *general magic square of order* n if the sum of elements in each row, column, and main diagonal is the same. The sum is the *magic number*. A general magic square A of order n is a *magic square*, denoted by $\mathrm{MS}(n)$, if the entries of A are distinct. A magic square A of order n is *normal* if the entries of A are n^2 consecutive integers. Usually, the entry in position (i, j) of a matrix A is denoted by $a_{i,j}$.

Magic squares have been studied for 4000 years. The Loh-Shu magic square is the oldest known magic square; its invention is attributed to Fuh-Hic, the mythical founder of Chinese civilization [4]. A lot of work has been done on construction of magic squares; for more details, the interested reader may refer to [1,3–6,8,11], and the references therein.

Magic rectangles are a natural generalization of magic squares. An $m \times n$ *general magic rectangle* is an $m \times n$ array consisting of natural numbers such that each row sum is the same and each column sum is the same (the two constants differ if $m \neq n$). An $m \times n$ general magic rectangle is a *magic rectangle* if its mn entries are distinct. An $m \times n$ magic rectangle A is *normal* if the entries of A are mn consecutive integers. Harmuth [9,10] proved the following.

Lemma 1.1. *For $m, n > 1$, there exists a normal $m \times n$ magic rectangle if and only if $m \equiv n \pmod 2$ and $(m, n) \neq (2, 2)$.*

Given a matrix A and a positive integer d. Let A^{*d} denote the matrix obtained by raising each element of A to the dth power. The matrix A is a *d-multimagic square*, denoted by $\mathrm{MS}(n, d)$, if A^{*e} is an $\mathrm{MS}(n)$ for $1 \leq e \leq d$. Clearly, if A is normal, then A^{*e} cannot be normal for all positive integers $e \geq 2$. When $d = 2$, an $\mathrm{MS}(n, 2)$ is a *bimagic square*. An $m \times n$ (general) d-multimagic rectangle can be defined in a similar way.

It was shown by Lucas [14] that there is no $\mathrm{MS}(3, 2)$ and no normal $\mathrm{MS}(4, 2)$. The first normal bimagic square was published by Pfeffermann in 1891: it has order 8 [15,5]. The following can be found in [5].

Lemma 1.2. *There exists a normal $\mathrm{MS}(n, 2)$ for $8 \leq n \leq 64$ and there is no normal $\mathrm{MS}(n, 2)$ for $n = 3, 4$.*

Recently, Derksen et al. [8] have provided a constructive procedure to make a large class of d-multimagic squares for each positive integer $d \geq 2$. For example, they proved the following.

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Lemma 1.3. *There exists a normal* MS(n^2, 2) *for all odd* $n \geq 3$.

A magic square is *pandiagonal* if the sum of elements in each broken diagonal is the magic number. A family of normal pandiagonal bimagic squares was given in [7,12]. In this paper, we shall provide a new construction of bimagic squares by means of a magic pair of orthogonal general bimagic squares. As its application, the following results are obtained.

Theorem 1.4. *There exists a normal* MS(mn, 2) *for all positive integers* m, n *such that* $m, n \notin \{2, 3, 6\}$ *and* $m \equiv n \pmod 2$.

Theorem 1.5. *There exists a normal* MS($4m$, 2) *if and only if* $m \geq 2$.

2. Construction of a normal MS(n, 2)

An $n \times n$ matrix A with entries in a set T is a *balanced square* if each element of T appears n times in A. Two balanced squares A and B of order n over T_1 and T_2 are *orthogonal* if $\{(a_{i,j}, b_{i,j}) | 0 \leq i, j \leq n-1\} = T_1 \times T_2$. Given squares A and B, let $A * B$ denote the "pointwise product", with $a_{i,j}b_{i,j}$ in position (i, j). Squares A and B form a *magic pair* if $A * B$ is a general magic square.

Let I_n be the set of nonnegative integers less than n, i.e., $I_n = \{0, 1, \ldots, n-1\}$.

Construction 2.1. *Given* $n \times n$ *matrices* A *and* B *over* I_n, *let* $C = nA + B$. *Then the matrix* C *satisfies the following.*

(i) *C is a normal* MS(n) *if A and B are a pair of orthogonal general* MS(n).
(ii) *C is a normal* MS(n, 2) *if A and B are a magic pair of orthogonal general* MS(n, 2).

Proof. (i) Since A and B are orthogonal, we have

$$\{(a_{i,j}, b_{i,j}) | 0 \leq i, j \leq n-1\} = I_n \times I_n,$$

which indicates that

$$\{c_{i,j} | 0 \leq i, j \leq n-1\} = \{na_{i,j} + b_{i,j} | 0 \leq i, j \leq n-1\} = I_{n^2}.$$

By hypothesis, A and B are both general magic squares. Suppose that S_A and S_B are the magic sum of A and B, respectively. We have

$$\sum_{i=0}^{n-1} c_{i,j} = \sum_{i=0}^{n-1} (na_{i,j} + b_{i,j}) = nS_A + S_B, \quad 0 \leq j \leq n-1,$$
$$\sum_{j=0}^{n-1} c_{i,j} = \sum_{j=0}^{n-1} (na_{i,j} + b_{i,j}) = nS_A + S_B, \quad 0 \leq i \leq n-1,$$
$$\sum_{i=0}^{n-1} c_{i,i} = \sum_{i=0}^{n-1} (na_{i,i} + b_{i,i}) = nS_A + S_B,$$
$$\sum_{i=0}^{n-1} c_{i,n-1-i} = \sum_{i=0}^{n-1} (na_{i,n-1-i} + b_{i,n-1-i}) = nS_A + S_B.$$

Thus C is a normal MS(n).

(ii) Since A and B are a magic pair of orthogonal general MS(n, 2) over I_n, by (i) C is a normal MS(n). Let $D = A * B$, and let $S_{A^{*2}}$, $S_{B^{*2}}$ and S_D be the magic sums of A^{*2}, B^{*2}, and D, respectively. For $i \in I_n$, we have

$$\sum_{j=0}^{n-1} c_{i,j}^2 = \sum_{j=0}^{n-1} (na_{i,j} + b_{i,j})^2 = \sum_{j=0}^{n-1} (n^2 a_{i,j}^2 + 2na_{i,j}b_{i,j} + b_{i,j}^2) = n^2 S_{A^{*2}} + 2nS_D + S_{B^{*2}}.$$

For $j \in I_n$, we have

$$\sum_{i=0}^{n-1} c_{i,j}^2 = \sum_{i=0}^{n-1} (na_{i,j} + b_{i,j})^2 = \sum_{i=0}^{n-1} (n^2 a_{i,j}^2 + 2na_{i,j}b_{i,j} + b_{i,j}^2) = n^2 S_{A^{*2}} + 2nS_D + S_{B^{*2}}.$$
$$\sum_{i=0}^{n-1} c_{i,i}^2 = \sum_{i=0}^{n-1} (na_{i,i} + b_{i,i})^2 = \sum_{i=0}^{n-1} (n^2 a_{i,i}^2 + 2na_{i,i}b_{i,i} + b_{i,i}^2) = n^2 S_{A^{*2}} + 2nS_D + S_{B^{*2}}.$$
$$\begin{aligned}\sum_{i=0}^{n-1} c_{i,n-1-i}^2 &= \sum_{i=0}^{n-1} (na_{i,n-1-i} + b_{i,n-1-i})^2 = \sum_{i=0}^{n-1} (n^2 a_{i,n-1-i}^2 + 2na_{i,n-1-i}b_{i,n-1-i} + b_{i,n-1-i}^2) \\ &= n^2 S_{A^{*2}} + 2nS_D + S_{B^{*2}}.\end{aligned}$$

Thus, C is a normal MS(n, 2). The proof is complete. □

3. Proof of Theorem 1.4

By Construction 2.1, to obtain a normal MS(mn), it suffices to find a magic pair of orthogonal general MS(mn, 2) over I_{mn}. In this section, we shall construct a magic pair of orthogonal general MS(mn, 2) over I_{mn} by means of orthogonal diagonal latin squares and rectangles.

A *latin square* of order n, denoted by LS(n), is an $n \times n$ array over an n-set S such that each element in S occurs exactly once in each row and exactly once in each column. A *transversal* in a latin square of order n is a set of n cells, one from each row and column, containing each of n elements exactly once. A latin square of order n is *diagonal* if its two main diagonals are both transversals. The following can be found in [2].

Lemma 3.1. *There exists a pair of orthogonal diagonal* LS(n) *if and only if* $n \neq 2, 3, 6$.

It is easy to see that a latin square must be balanced and that any diagonal latin square is also a general bimagic square. Therefore, a magic pair of orthogonal diagonal LS(n) is a magic pair of orthogonal general MS(n, 2). Thus, by Construction 2.1 we have the following corollary, which can also be found in [16].

Corollary 3.2. *If there exists a magic pair of orthogonal diagonal* LS(n) *over* I_n*, then there exists a normal* MS(n, 2).

By Corollary 3.2, to construct a normal MS(n, 2), it suffices to find a magic pair of orthogonal diagonal LS(n). Modifying the proof of Lemma 2.1 in [13], we have the following.

Lemma 3.3. *There exists a magic pair of orthogonal diagonal* LS(mn) *for all positive integers* m, n *such that* $m, n \notin \{2, 3, 6\}$ *and* $m \equiv n \pmod 2$.

Proof. By Lemma 3.1, we can suppose that A and B are orthogonal diagonal LS(m) over I_m, C and D are orthogonal diagonal LS(n) over I_n. Clearly, the sum of the elements in each row, column, and main diagonal of A is $m(m-1)/2$, and the sum of the elements in each row, column and main diagonal of C is $n(n-1)/2$.

By Lemma 1.1, we can suppose that H is an $m \times n$ magic rectangle over I_{mn}. Let S_r and S_c be the row sum and the column sum of H, respectively. It is easy to calculate that

$$S_r = \frac{1}{m}\sum_{h \in I_{mn}} h = \frac{n(mn-1)}{2}, \qquad S_c = \frac{1}{n}\sum_{h \in I_{mn}} h = \frac{m(mn-1)}{2}.$$

Let

$$E = (e_{i,j}), \qquad F = (f_{i,j}),$$

where

$$\begin{aligned} &e_{i,j} = a_{u,v} + mc_{s,t}, \qquad f_{i,j} = h_{b_{u,v},d_{s,t}}, \\ &i = u + sm, \qquad j = v + tm, \qquad 0 \le u, \qquad v \le m-1, \qquad 0 \le s, \qquad t \le n-1. \end{aligned}$$

By the proof of Lemma 2.1 in [13], E and F are a pair of orthogonal diagonal LS(mn) over I_{mn}. We now prove that E and F are also a magic pair.

For each $i \in I_{mn}$, we can write $i = u + sm,\ 0 \le u \le m-1,\ 0 \le s \le n-1$.

$$\begin{aligned} \sum_{0\le j\le mn-1} e_{i,j}f_{i,j} &= \sum_{0\le v\le m-1}\sum_{0\le t\le n-1} (a_{u,v} + mc_{s,t})h_{b_{u,v},d_{s,t}} \\ &= \sum_{0\le v\le m-1} a_{u,v} \sum_{0\le t\le n-1} h_{b_{u,v},d_{s,t}} + m\sum_{0\le t\le n-1} c_{s,t}\sum_{0\le v\le m-1} h_{b_{u,v},d_{s,t}} \\ &= \sum_{0\le v\le m-1} a_{u,v}S_r + m\sum_{0\le t\le n-1} c_{s,t}S_c \\ &= \frac{m(m-1)}{2}\frac{n(mn-1)}{2} + m\frac{n(n-1)}{2}\frac{m(mn-1)}{2} \\ &= \frac{mn(mn-1)^2}{4}, \end{aligned}$$

noting that $\{d_{s,t} | 0 \le t \le n-1\} = I_n$ for given $s \in I_n$ and $\{b_{u,v} | 0 \le v \le m-1\} = I_m$ for given $u \in I_m$.

Similarly, one can prove that for each $j \in I_{mn}$,

$$\sum_{0\le i\le mn-1} e_{i,j}f_{i,j} = \frac{mn(mn-1)^2}{4},$$

and

$$\sum_{0\le i\le mn-1} e_{i,i}f_{i,i} = \frac{mn(mn-1)^2}{4},$$

$$\sum_{0\le i\le mn-1} e_{i,mn-1-i}f_{i,mn-1-i} = \frac{mn(mn-1)^2}{4}.$$

Thus, E and F are a magic pair of orthogonal diagonal LS(mn). □

The proof of Theorem 1.4 now follows by combining Lemma 3.3 and Corollary 3.2.

4. Proof of Theorem 1.5

Let p and q be two positive integers such that $p, q \notin \{1, 3\}$. By Theorem 1.4, there exists a normal MS($4pq$, 2), which gives a partial result on the existence of normal bimagic squares of order $4m$.

In this section, we shall show that a normal MS($4m$, 2) exists for all positive integers $m \notin \{1, 3\}$. By Construction 2.1, to obtain a MS($4m$, 2), we need only to construct a magic pair of orthogonal general MS($4m$, 2). To do this, we will take advantage of idempotent self-orthogonal latin squares and magic rectangles.

A latin square X of order n over I_n is *idempotent* if $x_{i,i} = i$ for all $i \in I_n$. A latin square X is *self-orthogonal* if it is orthogonal to its transpose X^T. The following can be found in [2].

Lemma 4.1. *There exists an idempotent self-orthogonal* LS(n) *for all positive integers* $n \neq 2, 3, 6$.

Let m be a positive integer such that $m \notin \{1, 3\}$, and let $n = 2m$. By Lemmas 4.1 and 1.1, we may assume that X is an idempotent self-orthogonal latin square of order n over I_n and that H is a $2 \times n$ magic rectangle over I_{2n}, with rows and columns labeled with I_2 and I_n. Let S_r and S_c be the row sum and the column sum of H, respectively. The following is clear.

$$\sum_{j=0}^{n-1} h_{i,j} = n(2n-1)/2 = S_r, \quad i = 0, 1 \tag{1}$$

$$h_{0,j} + h_{1,j} = 2n - 1 = S_c, \quad j = 0, 1, \ldots, n-1 \tag{2}$$

from which we compute

$$\sum_{j=0}^{n-1} h_{0,j}^2 - \sum_{j=0}^{n-1} h_{1,j}^2 = \sum_{j=0}^{n-1}(h_{0,j}+h_{1,j})(h_{0,j}-h_{1,j}) = \sum_{j=0}^{n-1} S_c(h_{0,j}-h_{1,j})$$
$$= S_c\left(\sum_{j=0}^{n-1} h_{0,j} - \sum_{j=0}^{n-1} h_{1,j}\right) = S_c(S_r - S_r) = 0.$$

That is

$$\sum_{j=0}^{n-1} h_{0,j}^2 = \sum_{j=0}^{n-1} h_{1,j}^2 = n(2n-1)(4n-1)/6 = S_r^{(2)}. \tag{3}$$

It follows that

$$\sum_{j=0}^{n-1} h_{0,j}h_{1,j} = \sum_{j=0}^{n-1} h_{0,j}(S_c - h_{0,j}) = S_c\sum_{j=0}^{n-1} h_{0,j} - \sum_{j=0}^{n-1} h_{0,j}^2 = S_cS_r - S_r^{(2)}. \tag{4}$$

Letting $T_k = \{h_{k,j} | 0 \le j \le n-1\}$ for $k \in \{0, 1\}$, we define $n \times n$ matrices A_k and B_k as follows.

$$A_k = (a_{i,j}^{(k)}), \quad a_{i,j}^{(k)} = h_{k,x_{i,j}}, \ 0 \le i, j \le n-1.$$

$$B_k = (b_{i,j}^{(k)}), \quad b_{i,j}^{(k)} = \begin{cases} h_{1-k,x_{j,i}+1}, & \text{if } x_{j,i} \equiv 0 \pmod 2, \\ h_{1-k,x_{j,i}-1}, & \text{if } x_{j,i} \equiv 1 \pmod 2. \end{cases}$$

Clearly, A_k and B_k are two latin squares over T_k and T_{1-k}, respectively. It is not difficult to show that A_k and $B_{k'}$ are orthogonal for $k, k' \in \{0, 1\}$. In fact, if there exist $i_1, j_1, i_2, j_2 \in I_n$ such that $(a_{i_1,j_1}^{(k)}, b_{i_1,j_1}^{(k')}) = (a_{i_2,j_2}^{(k)}, b_{i_2,j_2}^{(k')})$, then we have

$$a_{i_1,j_1}^{(k)} = a_{i_2,j_2}^{(k)} \tag{5}$$

and

$$b^{(k')}_{i_1,j_1} = b^{(k')}_{i_2,j_2}. \tag{6}$$

From (5), we have $h_{k,x_{i_1,j_1}} = h_{k,x_{i_2,j_2}}$, and hence $x_{i_1,j_1} = x_{i_2,j_2}$. From (6), we can show that $x_{j_1,i_1} - x_{j_2,i_2} \equiv 0 \pmod 2$. Otherwise, we have $h_{1-k',x_{j_1,i_1}\pm 1} = h_{1-k',x_{j_2,i_2}\mp 1}$. It follows that $x_{j_1,i_1} \pm 1 = x_{j_2,i_2} \mp 1$, hence, $x_{j_1,i_1} - x_{j_2,i_2} \equiv 0 \pmod 2$, a contradiction. So we have $h_{1-k',x_{j_1,i_1}\pm 1} = h_{1-k',x_{j_2,i_2}\pm 1}$, which implies $x_{j_1,i_1} = x_{j_2,i_2}$. Note that X is an idempotent self-orthogonal latin square, we have $i_1 = i_2, j_1 = j_2$, which indicates that A_k and $B_{k'}$ are orthogonal.

Construct two $2n \times 2n$ matrices A and B as follows,

$$A = \begin{pmatrix} A_0 & A_0 \\ A_1 & A_1 \end{pmatrix}, \qquad B = \begin{pmatrix} B_0 & B_1 \\ B_0 & B_1 \end{pmatrix}. \tag{7}$$

Then we have the following.

Lemma 4.2. *If A and B are defined as in (7), then we have the following.*

(i) *A and B are a pair of orthogonal $2n \times 2n$ general bimagic rectangles over I_{2n}.*
(ii) *$D = (a_{i,j}b_{i,j})$ is a $2n \times 2n$ general rectangle over I_{2n}.*

Proof. (i) Clearly, $T_0 \cup T_1 = I_{2n}$. Since A_k and $B_{k'}$ are orthogonal for all $k, k' \in \{0, 1\}$ from the above discussion, we have

$$\begin{aligned}\{(a_{i,j}, b_{i,j}) | 0 \le i, j \le 2n-1\} &= \bigcup_{k,k' \in \{0,1\}} \{(a^{(k)}_{i,j}, b^{(k')}_{i,j}) | 0 \le i, j \le n-1\} \\ &= \bigcup_{k,k' \in \{0,1\}} (T_k \times T_{1-k'}) = I_{2n} \times I_{2n}.\end{aligned}$$

So, A and B are orthogonal over I_{2n}.

Let $S^{(1)}_r = S_r = n(2n-1)/2$, $S^{(2)}_r = n(2n-1)(4n-1)/6$. For each $i \in I_{2n}$, we can write $i = kn+s$, where $k \in \{0, 1\}$, $s \in I_n$. By (1) and (3), for $e \in \{1, 2\}$, we have

$$\sum_{j=0}^{2n-1} a^e_{i,j} = 2\sum_{j=0}^{n-1} (a^{(k)}_{s,j})^e = 2\sum_{j=0}^{n-1} h^e_{k,x_{s,j}} = 2S^{(e)}_r.$$

For each $j \in I_{2n}$, we can write $j = k'n + t$, where $k' \in \{0, 1\}$, $t \in I_n$. We have

$$\sum_{i=0}^{2n-1} a^e_{i,j} = \sum_{i=0}^{n-1} (a^{(0)}_{i,t})^e + \sum_{i=0}^{n-1} (a^{(1)}_{i,t})^e = \sum_{i=0}^{n-1} h^e_{0,x_{i,t}} + \sum_{i=0}^{n-1} h^e_{1,x_{i,t}} = 2S^{(e)}_r.$$

Thus, A is a $2n \times 2n$ general bimagic rectangle over I_{2n}. Similarly, one can prove that B is also a $2n \times 2n$ bimagic rectangle over I_{2n}, the sum of elements in each row or column of B^{*e} is also $2S^{(e)}_r$ for $e \in \{1, 2\}$.

(ii) For each $i \in I_{2n}$, we can write $i = kn + s$, where $k \in \{0, 1\}$, $s \in I_n$. We have

$$\begin{aligned}\sum_{j=0}^{2n-1} a_{i,j}b_{i,j} &= \sum_{j=0}^{n-1} a^{(k)}_{s,j} b^{(0)}_{s,j} + \sum_{j=0}^{n-1} a^{(k)}_{s,j} b^{(1)}_{s,j} \\ &= \sum_{j=0}^{n-1} a^{(k)}_{s,j} (b^{(0)}_{s,j} + b^{(1)}_{s,j}) = \sum_{j=0}^{n-1} a^{(k)}_{s,j} S_c = S_r S_c.\end{aligned}$$

For each $j \in I_{2n}$, we can also write $j = k'n + t$, where $k' \in \{0, 1\}$, $t \in I_n$. We have

$$\begin{aligned}\sum_{i=0}^{2n-1} a_{i,j}b_{i,j} &= \sum_{i=0}^{n-1} a^{(0)}_{i,t} b^{(k')}_{i,t} + \sum_{i=0}^{n-1} a^{(1)}_{i,t} b^{(k')}_{i,t} \\ &= \sum_{i=0}^{n-1} (a^{(0)}_{i,t} + a^{(1)}_{i,t}) b^{(k')}_{i,t} = \sum_{i=0}^{n-1} S_c b^{(k')}_{i,t} = S_r S_c.\end{aligned}$$

Thus, D is a $2n \times 2n$ general rectangle over I_{2n}. The proof is complete. □

One may hope that A and B are a magic pair of orthogonal general MS$(2n, 2)$ over I_{2n}. Unfortunately, the constructions given above cannot guarantee this property. To see this, we give an example.

Example 4.3. Let $m = 2, n = 4$,

$$H = \begin{pmatrix} 0 & 6 & 5 & 3 \\ 7 & 1 & 2 & 4 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 3 & 1 & 2 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 1 \\ 1 & 2 & 0 & 3 \end{pmatrix}.$$

It is easy to see that H is a 2×4 magic rectangle over I_8, $S_r = 14$, $S_c = 7$, $S_r^{(2)} = 70$ and X is an idempotent self-orthogonal LS(4) over I_4. By the above constructions, we have

$$A_0 = \begin{pmatrix} 0 & 3 & 6 & 5 \\ 5 & 6 & 3 & 0 \\ 3 & 0 & 5 & 6 \\ 6 & 5 & 0 & 3 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 7 & 4 & 1 & 2 \\ 2 & 1 & 4 & 7 \\ 4 & 7 & 2 & 1 \\ 1 & 2 & 7 & 4 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 1 & 4 & 2 & 7 \\ 2 & 7 & 1 & 4 \\ 7 & 2 & 4 & 1 \\ 4 & 1 & 7 & 2 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 6 & 3 & 5 & 0 \\ 5 & 0 & 6 & 3 \\ 0 & 5 & 3 & 6 \\ 3 & 6 & 0 & 5 \end{pmatrix},$$

$$A = \begin{pmatrix} A_0 & A_0 \\ A_1 & A_1 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 6 & 5 & 0 & 3 & 6 & 5 \\ 5 & 6 & 3 & 0 & 5 & 6 & 3 & 0 \\ 3 & 0 & 5 & 6 & 3 & 0 & 5 & 6 \\ 6 & 5 & 0 & 3 & 6 & 5 & 0 & 3 \\ 7 & 4 & 1 & 2 & 7 & 4 & 1 & 2 \\ 2 & 1 & 4 & 7 & 2 & 1 & 4 & 7 \\ 4 & 7 & 2 & 1 & 4 & 7 & 2 & 1 \\ 1 & 2 & 7 & 4 & 1 & 2 & 7 & 4 \end{pmatrix},$$

$$B = \begin{pmatrix} B_0 & B_1 \\ B_0 & B_1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 & 7 & 6 & 3 & 5 & 0 \\ 2 & 7 & 1 & 4 & 5 & 0 & 6 & 3 \\ 7 & 2 & 4 & 1 & 0 & 5 & 3 & 6 \\ 4 & 1 & 7 & 2 & 3 & 6 & 0 & 5 \\ 1 & 4 & 2 & 7 & 6 & 3 & 5 & 0 \\ 2 & 7 & 1 & 4 & 5 & 0 & 6 & 3 \\ 7 & 2 & 4 & 1 & 0 & 5 & 3 & 6 \\ 4 & 1 & 7 & 2 & 3 & 6 & 0 & 5 \end{pmatrix}.$$

By Lemma 4.2, A and B are a pair of orthogonal 8×8 general bimagic rectangles over I_8, but not a magic pair. In fact, for $D = (d_{i,j}) = (a_{i,j}b_{i,j})$, we have

$$\sum_{i=0}^{7} d_{i,i} = \sum_{i=0}^{7} a_{i,i}b_{i,i} = 136, \qquad \sum_{i=0}^{7} d_{i,7-i} = \sum_{i=0}^{7} a_{i,7-i}b_{i,7-i} = 60.$$

However, we can obtain a magic pair of orthogonal general MS$(2n, 2)$ over I_{2n} from A and B by doing some column and row permutations to A and B together according to the following three steps.

Suppose that A and B are defined as in (7). Let $\pi_1 = (n, 2n-1)(n+1, 2n-2)\cdots(n+\frac{n-2}{2}, n+\frac{n}{2})$ and $\pi_2 = (1, 2n-2)(3, 2n-4)\cdots(n-1, n)$ be two permutations on I_{2n}.

Step 1. Do the column permutation π_1 to A and B to get E and F, respectively,

$$E = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}, \qquad F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix},$$

where $E_k = (e_{i,j}^{(k)})$, $F_k = (f_{i,j}^{(k)})$, $k = 1, 2, 3, 4$, and for $i, j \in I_n$,

$$e_{i,j}^{(1)} = a_{i,j}^{(0)}, \qquad e_{i,j}^{(2)} = a_{i,n-1-j}^{(0)}, \qquad e_{i,j}^{(3)} = a_{i,j}^{(1)}, \qquad e_{i,j}^{(4)} = a_{i,n-1-j}^{(1)},$$

and

$$f_{i,j}^{(1)} = b_{i,j}^{(0)}, \qquad f_{i,j}^{(2)} = b_{i,n-1-j}^{(1)}, \qquad f_{i,j}^{(3)} = b_{i,j}^{(0)}, \qquad f_{i,j}^{(4)} = b_{i,n-1-j}^{(1)}.$$

Step 2. Do the row permutation π_1 to E and F to get M and N, respectively,

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}, \qquad N = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix},$$

where $M_k = (m_{i,j}^{(k)})$, $N_k = (n_{i,j}^{(k)})$, $k = 1, 2, 3, 4$. For $i, j \in I_n$,

$$m_{i,j}^{(1)} = e_{i,j}^{(1)} = a_{i,j}^{(0)}, \qquad m_{i,j}^{(2)} = e_{i,j}^{(2)} = a_{i,n-1-j}^{(0)},$$
$$m_{i,j}^{(3)} = e_{n-1-i,j}^{(3)} = a_{n-1-i,j}^{(1)}, \qquad m_{i,j}^{(4)} = e_{n-1-i,j}^{(4)} = a_{n-1-i,n-1-j}^{(1)},$$

and

$$n_{i,j}^{(1)} = f_{i,j}^{(1)} = b_{i,j}^{(0)}, \qquad n_{i,j}^{(2)} = e_{i,j}^{(2)} = b_{i,n-1-j}^{(1)},$$
$$n_{i,j}^{(3)} = f_{n-1-i,j}^{(3)} = b_{n-1-i,j}^{(0)}, \qquad n_{i,j}^{(4)} = f_{n-1-i,j}^{(4)} = b_{n-1-i,n-1-j}^{(1)}.$$

Step 3. Do the column permutation π_2 to M and N to get U and V, respectively,

$$U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}, \qquad V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix},$$

where $U_k = (u_{i,j}^{(k)})$, $V_k = (v_{i,j}^{(k)})$, $k = 1, 2, 3, 4$. For $i, j \in I_n$,

$$u_{i,j}^{(1)} = \begin{cases} m_{i,j}^{(1)}, & j \equiv 0 \pmod 2 \\ m_{i,n-1-j}^{(2)}, & j \equiv 1 \pmod 2 \end{cases} = a_{i,j}^{(0)},$$

$$u_{i,j}^{(2)} = \begin{cases} m_{i,n-1-j}^{(1)}, & j \equiv 0 \pmod 2 \\ m_{i,j}^{(2)}, & j \equiv 1 \pmod 2 \end{cases} = a_{i,n-1-j}^{(0)},$$

$$u_{i,j}^{(3)} = \begin{cases} m_{i,j}^{(3)}, & j \equiv 0 \pmod 2 \\ m_{i,n-1-j}^{(4)}, & j \equiv 1 \pmod 2 \end{cases} = a_{n-1-i,j}^{(1)},$$

$$u_{i,j}^{(4)} = \begin{cases} m_{i,n-1-j}^{(3)}, & j \equiv 0 \pmod 2 \\ m_{i,j}^{(4)}, & j \equiv 1 \pmod 2 \end{cases} = a_{n-1-i,n-1-j}^{(1)},$$

and

$$v_{i,j}^{(1)} = \begin{cases} n_{i,j}^{(1)}, & j \equiv 0 \pmod 2 \\ n_{i,n-1-j}^{(2)}, & j \equiv 1 \pmod 2 \end{cases} = \begin{cases} b_{i,j}^{(0)}, & j \equiv 0 \pmod 2, \\ b_{i,j}^{(1)}, & j \equiv 1 \pmod 2, \end{cases}$$

$$v_{i,j}^{(2)} = \begin{cases} n_{i,n-1-j}^{(1)}, & j \equiv 0 \pmod 2 \\ n_{i,j}^{(2)}, & j \equiv 1 \pmod 2 \end{cases} = \begin{cases} b_{i,n-1-j}^{(0)}, & j \equiv 0 \pmod 2, \\ b_{i,n-1-j}^{(1)}, & j \equiv 1 \pmod 2, \end{cases}$$

$$v_{i,j}^{(3)} = \begin{cases} n_{i,j}^{(3)}, & j \equiv 0 \pmod 2 \\ n_{i,n-1-j}^{(4)}, & j \equiv 1 \pmod 2 \end{cases} = \begin{cases} b_{n-1-i,j}^{(0)}, & j \equiv 0 \pmod 2, \\ b_{n-1-i,j}^{(1)}, & j \equiv 1 \pmod 2, \end{cases}$$

$$v_{i,j}^{(4)} = \begin{cases} n_{i,n-1-j}^{(3)}, & j \equiv 0 \pmod 2 \\ n_{i,j}^{(4)}, & j \equiv 1 \pmod 2 \end{cases} = \begin{cases} b_{n-1-i,n-1-j}^{(0)}, & j \equiv 0 \pmod 2, \\ b_{n-1-i,n-1-j}^{(1)}, & j \equiv 1 \pmod 2. \end{cases}$$

We have the following.

Lemma 4.4. *If A and B are defined as in (7), then U and V listed above are a magic pair of orthogonal general* MS$(2n, 2)$ *over* I_{2n}.

Proof. Since U and V are obtained from A and B under the same row or column permutations, by Lemma 4.2, U and V are also a pair of orthogonal $2n \times 2n$ general bimagic rectangles over I_{2n}, the sum of elements in each row or column of U^{*e} and V^{*e} is $2S_r^{(e)}$, $e = 1, 2$. By the same reason, $U * V$ is a $2n \times 2n$ general magic rectangle, the sum of elements in each row or column of $U * V$ is S_rS_c.

For $e = 1, 2$, we have

$$\begin{aligned} \sum_{i=0}^{2n-1} u_{i,i}^e &= \sum_{i=0}^{n-1} (u_{i,i}^{(1)})^e + \sum_{i=0}^{n-1} (u_{i,i}^{(4)})^e = \sum_{i=0}^{n-1} (a_{i,i}^{(0)})^e + \sum_{i=0}^{n-1} (a_{n-1-i,n-1-i}^{(1)})^e \\ &= S_r^{(e)} + S_r^{(e)} = 2S_r^{(e)}, \end{aligned}$$

$$\begin{aligned} \sum_{i=0}^{2n-1} (u_{i,n-1-i})^e &= \sum_{i=0}^{n-1} (u_{i,n-1-i}^{(2)})^e + \sum_{i=0}^{n-1} (u_{i,n-1-i}^{(3)})^e = \sum_{i=0}^{n-1} (a_{i,i}^{(0)})^e + \sum_{i=0}^{n-1} (a_{n-1-i,n-1-i}^{(1)})^e \\ &= S_r^{(e)} + S_r^{(e)} = 2S_r^{(e)}. \end{aligned}$$

Thus, U is a general bimagic square. Similarly, one can show that V is also a general bimagic square.

On the other hand,

$$\begin{aligned}\sum_{i=0}^{2n-1} u_{i,i}v_{i,i} &= \sum_{i=0}^{n-1} u^{(1)}_{i,i}v^{(1)}_{i,i} + \sum_{i=0}^{n-1} u^{(4)}_{i,i}v^{(4)}_{i,i}\\ &= \sum_{\substack{0\le i\le n-1\\ i\equiv 0 \pmod 2}} a^{(0)}_{i,i}b^{(0)}_{i,i} + \sum_{\substack{0\le i\le n-1\\ i\equiv 1 \pmod 2}} a^{(0)}_{i,i}b^{(1)}_{i,i} + \sum_{\substack{0\le i\le n-1\\ i\equiv 0 \pmod 2}} a^{(1)}_{n-1-i,n-1-i}b^{(0)}_{n-1-i,n-1-i}\\ &\quad + \sum_{\substack{0\le i\le n-1\\ i\equiv 1 \pmod 2}} a^{(1)}_{n-1-i,n-1-i}b^{(1)}_{n-1-i,n-1-i}\\ &= \sum_{\substack{0\le i\le n-1\\ i\equiv 0 \pmod 2}} a^{(0)}_{i,i}b^{(0)}_{i,i} + \sum_{\substack{0\le i\le n-1\\ i\equiv 1 \pmod 2}} a^{(0)}_{i,i}b^{(1)}_{i,i} + \sum_{\substack{0\le i\le n-1\\ i\equiv 1 \pmod 2}} a^{(1)}_{i,i}b^{(0)}_{i,i} + \sum_{\substack{0\le i\le n-1\\ i\equiv 0 \pmod 2}} a^{(1)}_{i,i}b^{(1)}_{i,i}. \qquad (*)\end{aligned}$$

Noting that for each $i \in I_n$, $x_{i,i} = i$. By the definition of A_k and B_k, we have

$$a^{(k)}_{i,i} = h_{k,i}, \qquad b^{(k)}_{i,i} = \begin{cases} h_{1-k,i+1}, & \text{if } i \equiv 0 \pmod 2,\\ h_{1-k,i-1}, & \text{if } i \equiv 1 \pmod 2,\end{cases} \qquad k = 0, 1.$$

$(*)$ becomes

$$\begin{aligned}\sum_{i=0}^{2n-1} u_{i,i}v_{i,i} &= \sum_{\substack{0\le i\le n-1\\ i\equiv 0 \pmod 2}} h_{0,i}h_{1,i+1} + \sum_{\substack{0\le i\le n-1\\ i\equiv 1 \pmod 2}} h_{0,i}h_{0,i-1} + \sum_{\substack{0\le i\le n-1\\ i\equiv 1 \pmod 2}} h_{1,i}h_{1,i-1} + \sum_{\substack{0\le i\le n-1\\ i\equiv 0 \pmod 2}} h_{1,i}h_{0,i+1}\\ &= \sum_{\substack{0\le i\le n-1\\ i\equiv 0 \pmod 2}} h_{0,i}h_{1,i+1} + \sum_{\substack{0\le i\le n-1\\ i\equiv 0 \pmod 2}} h_{0,i+1}h_{0,i} + \sum_{\substack{0\le i\le n-1\\ i\equiv 0 \pmod 2}} h_{1,i+1}h_{1,i} + \sum_{\substack{0\le i\le n-1\\ i\equiv 0 \pmod 2}} h_{1,i}h_{0,i+1}\\ &= \sum_{\substack{0\le i\le n-1\\ i\equiv 0 \pmod 2}} (h_{0,i}+h_{1,i})(h_{1,i+1}+h_{0,i+1})\\ &= \sum_{\substack{0\le i\le n-1\\ i\equiv 0 \pmod 2}} S_c^2 = \frac{n}{2}(2n-1)^2 = S_rS_c.\end{aligned}$$

In a similar way, one can readily check that

$$\sum_{i=0}^{2n-1} u_{i,2n-1-i}v_{i,2n-1-i} = S_rS_c.$$

Thus, U and V are a magic pair of orthogonal general MS$(2n, 2)$ over I_{2n}. The proof is complete. □

We are now in a position to give the proof of Theorem 1.5. Let $X = 2nU + V$, then X is a normal MS$(2n, 2)$ by Construction 2.1 and Lemma 4.4, where $n = 2m, m \notin \{1, 3\}$. Combining with Lemma 1.2, the proof of Theorem 1.5 is obtained. □

To illustrate the above constructions, we provide an example below.

Example 4.5. Let $m = 2$, $n = 4$. Let A and B be the same as in Example 4.3, i.e.,

$$A = \begin{pmatrix} 0&3&6&5&0&3&6&5\\ 5&6&3&0&5&6&3&0\\ 3&0&5&6&3&0&5&6\\ 6&5&0&3&6&5&0&3\\ 7&4&1&2&7&4&1&2\\ 2&1&4&7&2&1&4&7\\ 4&7&2&1&4&7&2&1\\ 1&2&7&4&1&2&7&4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1&4&2&7&6&3&5&0\\ 2&7&1&4&5&0&6&3\\ 7&2&4&1&0&5&3&6\\ 4&1&7&2&3&6&0&5\\ 1&4&2&7&6&3&5&0\\ 2&7&1&4&5&0&6&3\\ 7&2&4&1&0&5&3&6\\ 4&1&7&2&3&6&0&5 \end{pmatrix}.$$

Let $\pi_1 = (4, 7)(5, 6)$ and $\pi_2 = (1, 6)(3, 4)$ be two permutations on I_8.

Step 1. Do the column permutation π_1 to A and B to get E and F, respectively,

$$E = \begin{pmatrix} 0 & 3 & 6 & 5 & 5 & 6 & 3 & 0 \\ 5 & 6 & 3 & 0 & 0 & 3 & 6 & 5 \\ 3 & 0 & 5 & 6 & 6 & 5 & 0 & 3 \\ 6 & 5 & 0 & 3 & 3 & 0 & 5 & 6 \\ 7 & 4 & 1 & 2 & 2 & 1 & 4 & 7 \\ 2 & 1 & 4 & 7 & 7 & 4 & 1 & 2 \\ 4 & 7 & 2 & 1 & 1 & 2 & 7 & 4 \\ 1 & 2 & 7 & 4 & 4 & 7 & 2 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 4 & 2 & 7 & 0 & 5 & 3 & 6 \\ 2 & 7 & 1 & 4 & 3 & 6 & 0 & 5 \\ 7 & 2 & 4 & 1 & 6 & 3 & 5 & 0 \\ 4 & 1 & 7 & 2 & 5 & 0 & 6 & 3 \\ 1 & 4 & 2 & 7 & 0 & 5 & 3 & 6 \\ 2 & 7 & 1 & 4 & 3 & 6 & 0 & 5 \\ 7 & 2 & 4 & 1 & 6 & 3 & 5 & 0 \\ 4 & 1 & 7 & 2 & 5 & 0 & 6 & 3 \end{pmatrix}.$$

Step 2. Do the row permutation π_1 to E and F to get M and N, respectively,

$$M = \begin{pmatrix} 0 & 3 & 6 & 5 & 5 & 6 & 3 & 0 \\ 5 & 6 & 3 & 0 & 0 & 3 & 6 & 5 \\ 3 & 0 & 5 & 6 & 6 & 5 & 0 & 3 \\ 6 & 5 & 0 & 3 & 3 & 0 & 5 & 6 \\ 1 & 2 & 7 & 4 & 4 & 7 & 2 & 1 \\ 4 & 7 & 2 & 1 & 1 & 2 & 7 & 4 \\ 2 & 1 & 4 & 7 & 7 & 4 & 1 & 2 \\ 7 & 4 & 1 & 2 & 2 & 1 & 4 & 7 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 4 & 2 & 7 & 0 & 5 & 3 & 6 \\ 2 & 7 & 1 & 4 & 3 & 6 & 0 & 5 \\ 7 & 2 & 4 & 1 & 6 & 3 & 5 & 0 \\ 4 & 1 & 7 & 2 & 5 & 0 & 6 & 3 \\ 4 & 1 & 7 & 2 & 5 & 0 & 6 & 3 \\ 7 & 2 & 4 & 1 & 6 & 3 & 5 & 0 \\ 2 & 7 & 1 & 4 & 3 & 6 & 0 & 5 \\ 1 & 4 & 2 & 7 & 0 & 5 & 3 & 6 \end{pmatrix}.$$

Step 3. Do the column permutation π_2 to M and N to get U and V, respectively,

$$U = \begin{pmatrix} 0 & 3 & 6 & 5 & 5 & 6 & 3 & 0 \\ 5 & 6 & 3 & 0 & 0 & 3 & 6 & 5 \\ 3 & 0 & 5 & 6 & 6 & 5 & 0 & 3 \\ 6 & 5 & 0 & 3 & 3 & 0 & 5 & 6 \\ 1 & 2 & 7 & 4 & 4 & 7 & 2 & 1 \\ 4 & 7 & 2 & 1 & 1 & 2 & 7 & 4 \\ 2 & 1 & 4 & 7 & 7 & 4 & 1 & 2 \\ 7 & 4 & 1 & 2 & 2 & 1 & 4 & 7 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 3 & 2 & 0 & 7 & 5 & 4 & 6 \\ 2 & 0 & 1 & 3 & 4 & 6 & 7 & 5 \\ 7 & 5 & 4 & 6 & 1 & 3 & 2 & 0 \\ 4 & 6 & 7 & 5 & 2 & 0 & 1 & 3 \\ 4 & 6 & 7 & 5 & 2 & 0 & 1 & 3 \\ 7 & 5 & 4 & 6 & 1 & 3 & 2 & 0 \\ 2 & 0 & 1 & 3 & 4 & 6 & 7 & 5 \\ 1 & 3 & 2 & 0 & 7 & 5 & 4 & 6 \end{pmatrix}.$$

Then by Lemma 4.4, U and V are a magic pair of orthogonal general MS(8, 2) over I_8. It is not difficult to calculate

$$\sum_{i=0}^{7} u_{i,i}v_{i,i} = 0\cdot 1 + 6\cdot 0 + 5\cdot 4 + 3\cdot 5 + 4\cdot 2 + 2\cdot 3 + 1\cdot 7 + 7\cdot 6 = 98,$$

$$\sum_{i=0}^{7} u_{i,7-i}v_{i,7-i} = 0\cdot 6 + 6\cdot 7 + 5\cdot 3 + 3\cdot 2 + 4\cdot 5 + 2\cdot 4 + 1\cdot 0 + 7\cdot 1 = 98.$$

Remark. It is readily checked that for integer $m \notin \{1, 3\}$, the normal MS$(4m, 2)X$ obtained from the proof of Theorem 1.5 having the following properties:

(I) $\sum_{i=0}^{2m-1} x_{i,j} = \sum_{i=2m}^{4m-1} x_{i,j} = (4m+1)S_r,\ 0 \le j \le 4m-1,$
(II) $\sum_{\substack{0\le j\le 4m-1\\ j\equiv 0 \pmod 2}} x_{i,j} = \sum_{\substack{0\le j\le 4m-1\\ j\equiv 1 \pmod 2}} x_{i,j} = (4m+1)S_r,$
(III) $x_{i,i} + x_{4m-1-i,4m-1-i} = x_{i,4m-1-i} + x_{4m-1-i,i} = (4m+1)S_c,$

where $S_r = m(4m-1)$ and $S_c = 4m-1$.

We should point out that a normal MS$(4m, 2)$ having the properties (I)–(III) will be useful in constructing sparse bimagic squares, which will be described in our next paper.

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